

New universality class in percolation on multifractal scale-free planar stochastic lattice

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We investigate site percolation on a weighted planar stochastic lattice (WPSL) which is a multifractal and whose dual is a scale-free network. Percolation is typically characterized by percolation threshold p_c and by a set of critical exponents β , γ , ν which describe the critical behavior of percolation probability $P(p) \sim (p_c - p)^\beta$, mean cluster size $S \sim (p_c - p)^{-\gamma}$ and the correlation length $\xi \sim (p_c - p)^{-\nu}$. Besides, the exponent τ characterizes the cluster size distribution function $n_s(p_c) \sim s^{-\tau}$ and the fractal dimension d_f the spanning cluster. We obtain an exact value for p_c and for all these exponents. Our results suggest that the percolation on WPSL belong to a new universality class as its exponents do not share the same value as for all the existing planar lattices.

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Percolation is perhaps one of the most studied problems in statistical physics because it provides a general framework of statistical theories that deal with structural and transport properties in porous or heterogeneous media [1, 2]. To study percolation one has to first choose a skeleton, namely an empty lattice or a graph. The model can then be defined by one sentence. Each site/bond of the lattice or graph is either occupied with probability p or remains empty with probability $1 - p$ independent of the state of its neighbors. For small values of p we see mostly single or a few contiguous occupied sites which are called clusters. As p increases, the mean cluster size always keeps growing at an increasingly faster rate till it comes to a state when suddenly a macroscopic cluster appears for the first time spanning from one end of the lattice to its opposite end. This sudden onset of a spanning cluster in an infinite system occurs at a particular value of p known as the percolation threshold p_c . This is accompanied by sudden or abrupt change in the behavior of the observable quantities with a small change in its control parameter p . Such change is almost always found to be the signature of phase transition that occur in a wide range of phenomena [3, 4]. This is why scientists in general and physicists in particular find percolation theory so attractive. Indeed, the insight into the percolation problem facilitates the understanding of phase transition and critical phenomena that lies at the heart of the modern development of statistical physics.

Percolation on disordered lattice is potentially of great interest since many real-life phenomena deal with such disordered systems [5]. In recent decades there has been a surge of research activities in studying percolation on random and scale-free network because the coordination number disorder of these networks is closely tied to many natural and man-made skeleton or medium through which percolation occurs. For instance, infectious diseases, computer viruses, opinion, rumors, etc spreads usually through networks [6–11]. Besides, flow of fluids usually takes place through porous medium or through rocks and hence the architecture of the skeleton is anything but regular [12, 13]. In fact, transport of fluid through multifractal porous media such as sedimentary strata and in oil reservoir is of great interest in

geological systems [14, 15]. In this rapid communication, we investigate percolation on weighted planar stochastic lattice (WPSL). One of us recently has shown that its coordination number distribution follows a power-law and its size distribution can be best described as multifractal [16]. In contrast, scale-free networks too have power-law coordination number distribution but nodes or sites in the scale-free networks are neither embedded on spatial positions nor have edges or surfaces. Our goal is to find how the two aspects, multifractality and power-law coordination number distribution, leave their signature in the percolation processes. Classification of percolation into universality classes depending on the common sets of critical exponents has been of wide interest. It is well-known and widely accepted that the values of the critical exponents depend only on the dimension of the lattice and independent of its detailed structure and of the type of percolation, namely site or bond percolation. It is note worthy to mention that when a planar lattice is only multifractal but its dual is not a scale-free network, the resulting percolation still belongs to the same universality as the one for regular planar lattice [17]. We report for the first time that all the exponents for WPSL are completely different from the existing known values for the planar lattice ($d = 2$) and hence the percolation on WPSL belongs to a new universality class.

The construction process of the WPSL starts with an initiator, say a square of unit area. The generator is then defined as the one that divides randomly first the initiator into four smaller blocks. Thereafter at each step the generator is applied to only one block, by picking it preferentially with respect to areas, and divides them randomly into four smaller blocks. Each step of the division process is defined as one time unit. It is thus a process of partitioning an initiator by random sequential division that creates contiguous blocks of different sizes. The number of blocks N after time t therefore is $N = 1 + 3t$ and hence it grows albeit the sum of the areas of all the blocks remains the same. Thus, the number of blocks N increases with time at the expense of the size of the blocks. Indeed, the mean cell area decreases with N like $\langle a \rangle \sim N^{-1}$ or with t like $\langle a \rangle \sim t^{-1}$ since $\langle a \rangle = a/N$ where the total area a of all the blocks is always equal

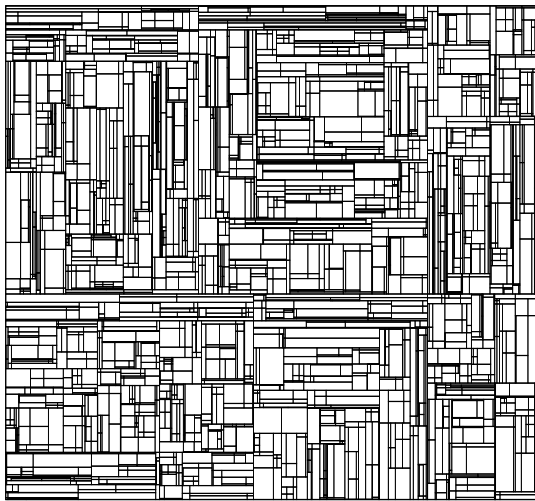


FIG. 1: A snapshot of the weighted stochastic lattice.

to one. We therefore need to emphasize two things here. First, we can define the size of the side L of the WPSL as $t^{1/2}$ like we do for square lattice $L = N^{1/2}$. Second, to make the cells of the WPSL have the same size, in the statistical sense, as we increase N , we have to scale up the cell sizes by a factor of t . Else, we just need to multiply each quantity we measure by a factor t to compensate the decreasing size of the blocks.

Perhaps, the construction process of WPSL is trivially simple but its various properties are far from simple. Firstly, it evolves following several non-trivial conservation laws, namely $\sum_i^N x_i^{n-1} y_i^{4/n-1}$ is independent of time or size of the lattice $\forall n$, where x_i and y_i are the length and width of the i th block. Secondly, its dual, obtained by replacing each block with a node at its center and common border between blocks with an edge joining the two vertices, emerges as a scale-free network [16]. Thirdly, if one considers that the i th block is populated with probability $p_i \sim x_i^3$ or y_i^3 then the q th moment of p_i can be shown to exhibit power-law $Z_q(\delta) \sim \delta^{-\tau(q)}$ where δ is the square root of the mean block area and

$$\tau(q) = \sqrt{9q^2 + 16} - (3q + 2). \quad (1)$$

Note that $\tau(0) = 2$ is the dimension of the WPSL and $\tau(1) = 0$ follows from the normalization of the probabilities $\sum p_i = 1$ [18]. The Legendre transform of $\tau(q)$ on the other hand gives the multifractal spectrum $f(\alpha)$ where the exponent α is the negative derive of $\tau(q)$ with respect to q . Yet another features of the WPSL is that it emerges through evolution and the area size distribution function of its cells exhibits dynamic scaling [19].

To study percolation on the WPSL we employ Ziff-Newman algorithm [20] in which all the labeled sites or cells $i = 1, 2, 3, \dots, (1 + 3t)$ are first randomized and arranged in an order in which the sites will be occupied. The good thing about this algorithm is that we can create percolation states consisting of $n + 1$ occupied sites

simply by occupying one more site to its immediate past percolation state consisting of n occupied sites. Each time thereafter we occupy a site, it may happen that either an isolated cluster is formed or a group of contiguous sites linked by common border may get bigger either by agglomeration or by coagulation. We keep track of the number of clusters and their sizes as a function of n *vis-a-vis* the occupation probability $p = n/(1 + 3t)$. In fact, the product of the number of occupied sites n and the mean area $1/(1 + 3t)$ is equal to the mean area of all the occupied sites $\langle a(n) \rangle$ which is equal to 1 when all the sites are occupied i.e., when $n = N$. In our simulation we use periodic boundary condition where the lattice is viewed as a torus, thus without edge or surface, where sites are randomly occupied with probability p .

In percolation, one of the primary objectives is to find the occupation probability p_c at which a cluster of contiguous occupied sites span the entire lattice, either horizontally or vertically, for the first time. Of course, the occupation probability at which it occurs at each independent realization on finite size lattice will not be the same. In reality, we can get spanning even at very much less than p_c or not get it even at a much higher p than p_c . This is exactly why the percolation theory is a part of statistical physics. One way of dealing with this is to use the idea of spanning probability $W(p)$ [21]. Consider that we have performed m independent realizations and for each realization we check exactly at what value of $p = n/N$ there appears a cluster that connects the two opposite ends either horizontally or vertically, whichever come first. The spanning probability $W(p)$ is the probability of occurrence of spanning cluster. It is obtained by finding the relative frequency of occurrence of spanning cluster out of m independent realizations. The plot in Fig. (2), shows $W(p)$ as a function of p for three different lattice sizes. It thus represents the probability of finding a spanning cluster at occupation probability p for a fixed lattice size. One interesting point is that all the three plots meet at one particular point. It has a special significance as it means that if we could have data for infinitely large lattice the resulting plot would also cross at the same meeting point. This meeting point is actually the percolation threshold $p_c = 0.526846$ for the WPSL.

A careful look at the plots of Fig. (2) we find that if we increase L then a given fixed value of W is obtained at increasingly higher value of p for $p < p_c$. To quantify this we draw a horizontal line, for instance at $W(p) = 0.3$, and a vertical line passing through the p_c value. Say, the horizontal line intersects all the three curves and the vertical line for different L at A, B, C and at O. We find that the distance OA, OB, OC etc which represents $(p_c - p)$ and plot them in the log-log scale as a function of t . The resulting plot gives a straight line with slope 0.2966 ± 0.0055 . Using $L \sim t^{1/2}$ we can write

$$(p_c - p) \sim L^{-1/\nu}, \quad (2)$$

where $1/\nu \sim 0.6$ or $\nu = 5/3$. This is different and quite a bit higher than the known value $\nu = 4/3$ for all planar

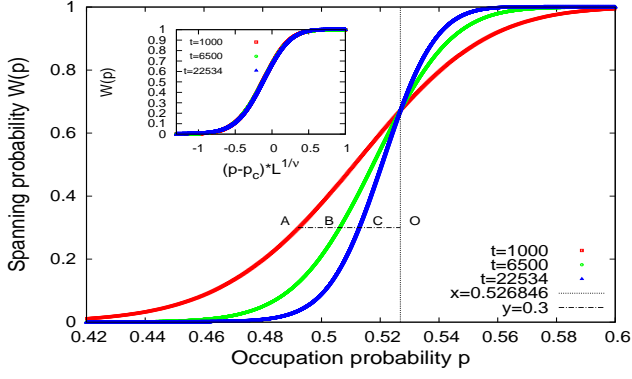


FIG. 2: Plots of spanning probability $W(p)$ vs occupation probability p for different lattice size. The vertical is drawn at $p_c = 0.526846$. In the inset, we plot $W(p)$ vs $(p - p_c)L^{1/\nu}$, where $\nu = 5/3$ and find an excellent data collapse.

lattices. For consistency check, one can now plot the p values at A, B, C etc versus $L^{-1/\nu}$. The intercept of the resulting linear fit gives the desired p_c value and hence this offers an alternative method of measuring p_c . The quantity $(p_c - p)L^{1/\nu}$ is a dimensionless quantity, according to Eq. (2), in the sense that for a given value of W as $L \rightarrow \infty$ the value of $(p_c - p) \rightarrow 0$ such that the numerical value of $(p_c - p)L^{1/\nu}$ remains invariant regardless of the lattice size L . We now plot $w(p)$ as a function of $(p_c - p)L^{1/\nu}$, see the inset of Fig. (2), and find that all the distinct curves of Fig. (3) collapse onto a single universal curve. It implies, according to finite size scaling hypothesis, that

$$W(p) \sim L^\eta \phi\left((p - p_c)L^{1/\nu}\right), \quad (3)$$

with exponent $\eta = 0$ where ϕ is the scaling function [22]. It states that the spanning probability W itself is a dimensionless quantity provided it is measured in the scaled variable $(p_c - p)L^{1/\nu}$ [23]. It also means that the spanning probability for infinite lattice size would be like a step function around p_c .

It is well-known that like Ising model percolation too display a continuous phase transition and hence like magnetization of the Ising model there must be an order parameter for the percolation model too. The fact is that not all the occupied sites belong to spanning cluster. We thus can define the percolation probability P which must be zero below p_c and should increase continuously beyond p_c - a characteristic feature for order parameter. We define it as the ratio of the area of the spanning cluster $A_{\text{span}}t$ to the total area of the lattice at and hence $P(p) = A_{\text{span}}/a$ since the total area of the lattice is always equal to one. Unlike $W(p)$ vs p the distinct curves of the $P(p)$ vs p plots, see Fig. (3), for different size do not meet at one unique value, namely at p_c which we can only appreciate if we zoom in. Nevertheless, following the same procedure we once again find $(p_c - p) \sim L^{-1/\nu}$ with the same ν value. Like for $W(p)$ if we plot P as a function $(p_c - p)L^{1/\nu}$ we do not get data

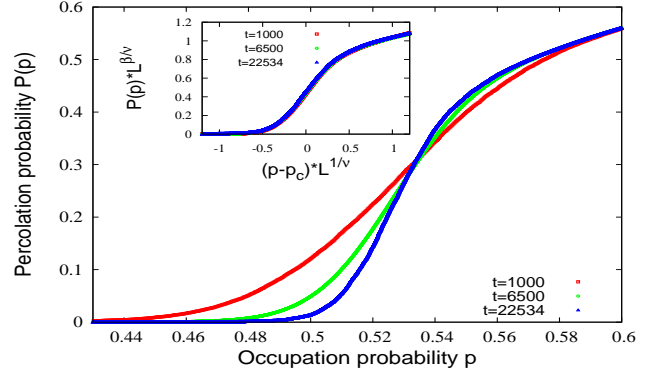


FIG. 3: Plots of percolation probability $P(p)$ vs p for three different size of the WPSL. In the inset we plot the same data but in the scaled variables $PL^{\beta/\nu}$ and $(p_c - p)L^{1/\nu}$ and find an excellent data-collapse.

collapse as before, instead we see that for a given value of $(p_c - p)L^{1/\nu}$ the P value decreases with lattice size L following a power-law

$$P \sim L^{-\beta/\nu}, \quad (4)$$

where $\beta/\nu = 0.135 \pm 0.0076$. It implies that for a given value of $(p_c - p)L^{1/\nu}$ the numerical value of $PL^{\beta/\nu}$ must remain invariant regardless of the lattice size of L . That is, if we now plot $PL^{\beta/\nu}$ vs $(p_c - p)L^{1/\nu}$ all the distinct plots of P vs p should collapse into a single universal curve. Indeed, such data-collapse is shown in the inset of Fig. (3) which implies that percolation probability P exhibits finite-size scaling

$$P(p_c - p, L) \sim L^{-a} \phi\left((p_c - p)L^{1/\nu}\right). \quad (5)$$

Now, eliminating L in favor of $p_c - p$ in Eq. (4) we get

$$P \sim (p_c - p)^\beta, \quad (6)$$

where $\beta \sim 0.225$ or $\beta = 9/40$ for WPSL whereas $\beta = 5/36$ for all other known planar lattices.

Percolation is all about clusters and hence the cluster size distribution function $n_s(p)$ plays a central role in the description of the percolation theory. It is defined as the number of clusters of size s per site. The quantity $sn_s(p)$ therefore is the probability that an arbitrary site belongs to a cluster of s sites and $\sum_{s=1}^\infty sn_s$ is probability that an arbitrary site belongs to a cluster of any size which is in fact equal to p . The mean cluster size $S(p)$ therefore is given by

$$S(p) = \sum_s s f_s = \frac{\sum_s s^2 n_s}{\sum_s s n_s}, \quad (7)$$

where the sum is over the finite clusters only. In the case of percolation on the WPSL, we regard s as the cluster area. It is important to mention that each time we evaluate the ratio of the second and the first moment of

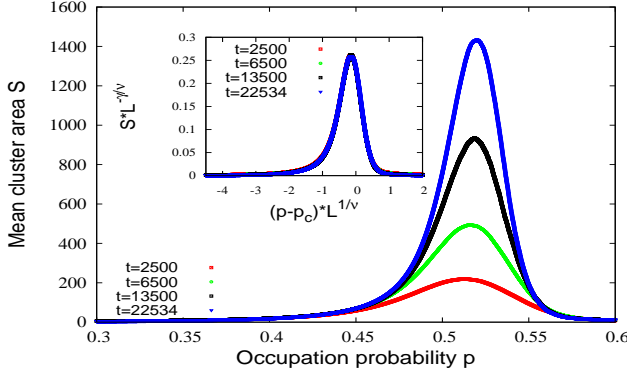


FIG. 4: Mean cluster area $S(p)$ as function of occupation probability p with different lattice size L . In the inset we plot the scaled variables $SL^{-\gamma/\nu}$ vs $(p-p_c)L^{1/\nu}$ and find that data for different system sizes are well collapsed in a single universal curve.

of n_s we also have to multiply the result by t , the time at which the snapshot of the lattice is taken, to compensate the decreasing block size with increasing block number N . The mean cluster size therefore is $S = \frac{1}{p} \sum_s s^2 n_s \times t$ where $\sum_s s n_s = p$ is the sum of the areas of all the clusters. Note that the spanning cluster is excluded from both the sums of Eq. (7). In Fig. (4) we plot $S(p)$ as a function of p for different lattice sizes L . We observe that there are two main effects as we increase the lattice size. First, we see that the mean cluster area always increases as we increase the occupation probability. However, as the p value approaches to p_c , we find that the peak height grows profoundly with L .

The increase of the peak height can be quantified by plotting these heights as a function of L in the log-log scale and find

$$S \sim L^{\gamma/\nu}, \quad (8)$$

where $\gamma/\nu = 1.73 \pm 0.006321$. A careful observation reveals that there is also a shift in the p value at which the peaks occur. We find that the magnitude of this shift $(p_c - p)$ becomes smaller with increasing L following a power-law $(p_c - p) \sim L^{-1/\nu}$. We now plot the same data in Fig (4) by measuring the mean cluster area S in unit of L^b and $(p_c - p)$ in unit of $L^{-1/\nu}$ respectively and find that all the distinct plots of S vs p collapse into one universal curve, see the inset of the same figure. It again implies that the mean cluster area too exhibits finite-size scaling

$$S \sim L^b \phi\left((p_c - p)L^{1/\nu}\right). \quad (9)$$

Eliminating L from Eq. (8) in favor of $(p_c - p)$ using $(p_c - p) \sim L^{-1/\nu}$ we find that the mean cluster area diverges

$$S \sim (p_c - p)^{-\gamma}, \quad (10)$$

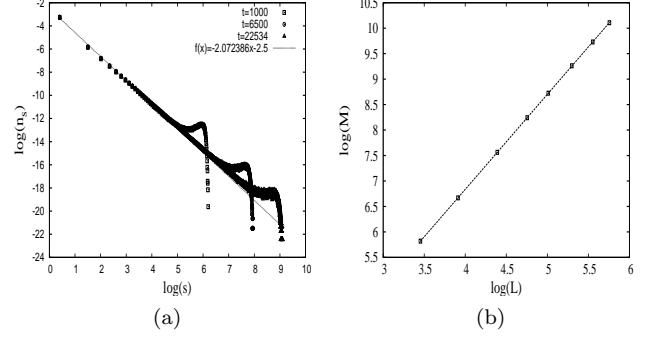


FIG. 5: (a) The plot of $\log(n_s)$ vs $\log(s)$ for different lattice size L . (b) The double-logarithmic plot of the size of the spanning cluster M against the lattice size L .

where $\gamma = 2.883$ which we can approximately write $\gamma = 173/60$. In contrast, $\gamma = 43/18$ for all other planar lattices.

We can also obtain the exponent τ by plotting the cluster area distribution function $n_s(p_c)$ at p_c . We plot it in the log-log scale and find a straight line except near the tail. However, we also observe that as the lattice size increases the extent up to which we get a straight line having the same slope increases. It implies that if we performed on WPSL of infinitely large size we would have a perfect straight line obeying $n_s(p_c) \sim s^{-\tau}$ with $\tau = 2.0724$ which is less than its value for all other known planar lattices $\tau = 187/91$. We already know that mean cluster area $S \rightarrow \infty$ as $p \rightarrow p_c$. According to Eq. (7), S can only diverge if its numerator diverges. Generally, we know that $\sum_{s=1}^{\infty} s^\alpha$ converges if $\alpha < -1$ and diverges if $\alpha \geq -1$. Applying it into both numerator and denominator of Eq. (7) at p_c gives a bound that $2 < \tau < 3$. Assuming

$$n_s(p) \sim s^{-\tau} e^{-s/s_\xi}, \quad (11)$$

and using it in Eq. (7) and taking continuum limit gives

$$S \sim s_\xi^{3-\tau}. \quad (12)$$

We know that s_ξ diverges like $(p_c - p)^{1/\sigma}$ where $\sigma = 1/(\nu d_f)$ and hence comparing it with Eq. (10) we get

$$\tau = 3 - \gamma\sigma. \quad (13)$$

Note that the ramified nature of the spanning cluster at p_c is reminiscent of fractal. Indeed, we find that the the fractal dimension d_f of the spanning cluster can be obtained by finding the gradient of the plot of the size of the spanning cluster M as a function of lattice size L in the log-log scale (see Fig (5b)). We find $d_f = 1.865$ which can also be written as $d_f = 373/200$ for WPSL and that for square, triangular, honeycomb, Voronoi lattices is $d_f = 93/48$. Using the value of γ and σ in Eq. (13) we get $\tau = 2.0724$ which we can approximately write as $773/373$. This is consistent with what we found from the slope of $\log[n_s(p_c)]$ vs $\log[s]$ plot shown in Fig. (5a).

Exponents	regular planar lattice	WPSL
ν	4/3	5/3
β	5/36	9/40
γ	43/18	173/60
τ	187/91	773/373
d_f	91/48	373/200

To summarize, we have studied percolation on a scale-free multifractal planar lattice. We obtained the p_c value and the characteristic exponents $\nu, \beta, \gamma, \tau, \sigma$ and d_f which characterize the percolation transition. Note that it is the sudden onset of a spanning cluster at the threshold p_c which is accompanied by discontinuity or divergence of some observable quantities at the threshold make the percolation transition a critical phenomena. One of the most interesting and useful aspects of percolation theory so far known is that the values of the various exponents depend only on the dimensionality of the lattice as they are found independent of the type of lattice (e.g., hexagonal, triangular or square, etc.) and the type of percolation (site or bond). This central property of percolation theory is known as universality. Recently, Corso *et al* per-

formed percolation on a particular multifractal planar lattice whose coordination number distribution is, however, not scale-free like WPSL and still they found the exponents as for all the planar regular lattices [17]. Thus the most expected result would be to find a different value for p_c value as its coordination number distribution is totally different than any known planar lattice. However, finding a complete different set of values, see the table, for all the characteristic exponents was not expected since WPSL too a planar lattice. Interestingly, like existing values for regular planar lattices, the exponents of the values for WPSL too satisfy the scaling relations $\beta = \nu(d - d_f)$, $\gamma = \nu(2d_f - d)$, $\tau = 1 + d/d_f$. We can this conclude that percolation on WPSL belongs to a new universality class. It would be interesting to check the role of the exponents γ of the power-law coordination number distribution in the classification of universality classes. We intend to do it in our future endeavour.

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